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PRESENTING THE FORMAL THEORY OF HIERARCHICAL COMPLEXITY

MICHAEL LAMPORT COMMONS
Department of Psychiatry, Harvard Medical School, Cambridge, Massachusetts, USA

ALEXANDER PEKKER
University of Texas at Austin, Austin, Texas, USA

The formal theory of the Model of Hierarchical Complexity is presented. Complexity theories generally exclude the concept of hierarchical complexity; Developmental Psychology has included it for over 20 years. It also applies to social systems and non-human systems. Formal axioms for the Model are outlined. The model assigns an order of hierarchical complexity to every task, using natural numbers, establishing a quantal notion of stage and stages of performance. This formalizes properties of stage theories in psychology. The formal theory of the model enables extending the concept of hierarchical complexity to any field where tasks and their performances exist.

KEYWORDS: Axioms, distributivity, formal theory, horizontal complexity, Model of Hierarchical Complexity, quantal, stage of performance.

This article presents the theoretical basis of the Model of Hierarchical Complexity in formal, axiomatic terms. The first article in this issue supplied the background and basic concepts of the Model; this presentation assumes that background. Sections that follow present the mathematical bases of actions and hierarchical complexity, the hierarchical complexity axioms and their consequences, the definition of stages, and the measure of hierarchical complexity. For more expansive discussion and reports of studies, including the use of Rasch (1980) analysis to test the theory and its predictions, see Commons, Goodheart et al. (2007).

MATHEMATICAL BASIS

The Model of Hierarchical Complexity is a mathematical model. It is useful to begin the presentation of the formal, axiomatic version of the theory with an illustration of the mathematical complexities that define the Model.

Consider the action of distributivity. Distributivity is the property of addition and multiplication on the real numbers that ensures that $a \times (b + c) = (a \times b) +$

Address correspondence to Michael Lamport Commons, Ph.D., Dare Institute, 234 Huron Ave., Cambridge, MA 02138-1328, USA. E-mail: commons@tiac.net

$(a \times c)$. Of course distributivity also plays a fundamental role in more general contexts, such as the complex numbers and the definition of rings in modern algebra. The distributive law serves as a motivation for a newer form of complexity, called hierarchical complexity, formally presented here.

The distributive law suggests that the task of evaluating $a \times (b + c)$ is more complex than the task of evaluating $(a + b) + c$ or even the two-part task of first evaluating $a + b$ and then evaluating $c \times d$. The evaluation of $(a + b) + c$ is no more complex than addition, performed either as $(a + b) + c$ or $a +$ $(b + c)$; the organization of the two actions of addition is arbitrary. Similarly, in the two-part task, evaluating $a + b$ and then $c \times d$ yields the same result as first evaluating $c \times d$ and then $a + b$. Both of these are chain actions. On the other hand, the evaluation of $a \times$ $(b + c)$ requires a non-arbitrary organization of addition and multiplication, or, equivalently, the distributive law, and is therefore more complex than addition or multiplication. In modern algebra, the non-arbitrary coordination of addition and multiplication leads to the definition of rings, and the expressions in ring theory are usually more complex than the expressions in group theory (which involve only one operation).

We refer to addition and multiplication as actions, a term that is commonly used by developmental psychologists to refer to events that produce outcomes, that is, accomplish certain tasks. The study of tasks appears in psychophysics (a branch of stimulus control theory in psychology) (Green and Swets, 1966; Luce, 1963) and in artificial intelligence (Goel and Chandrasekaran, 1992). Actions may be attributed to organisms, social groups, and computers. They may be combined to produce new, more complex actions (Binder, 2000). This article describes how to measure the complexity of an action and relates it to the complexity of other actions. The axioms presented in the sections that follow build on Piaget (e.g., Inhelder and Piaget, 1958) and his intellectual descendants (e.g., Campbell, 1991; Campbell and Bickhard, 1986; Tomasello and Farrar, 1986).

There are two types of complexity: horizontal (traditional) and vertical (hierarchical) (see "Introduction to the Model of Hierarchical Complexity," this issue). In traditional concepts of complexity, an action's complexity is determined by the number of times a specific subaction is repeated. In hierarchical complexity, the complexity of an action is determined by the non-arbitrary organization of subactions. Thus, the order of hierarchical complexity of an action is one greater than the order of hierarchical complexity of its subactions when they are organized in a non-arbitrary way.

To illustrate one difference between these types of complexity, consider the action A of evaluating $1 + 2$ and the action B of evaluating $(1 + 2) + 3$.[1] The traditional complexity of A is smaller than the traditional complexity of B because the action of addition is executed less often in A than in B. On the other hand, since A differs from B only in how many times addition is executed, but not in the organization of the addition, A and B have the same hierarchical complexity. This example shows that the two types of complexity are independent and incommensurate.

We begin by defining the fundamental terms. In a given system, there exist certain tasks that are to be accomplished. These tasks are accomplished via task-actions. Formally, *a task-action*, often abbreviated simply as an action, is defined inductively. There exists a unique *simple action* $\tilde{A}$, which is the simplest action possible in a system. This is in agreement with Luce's choice theory (Luce, 1959). Every other action A consists of at least two (and possibly infinitely many) previously defined actions and a *rule* for organizing those previously defined actions. Thus, every nonsimple action A is an ordered pair $A = (\{A_1, \ldots\}, R)$ where the first component is a multi-set of at least two previously defined actions A_i composing A and R is the rule for organizing those actions.

There are two categories of rules: chain rules and coordination rules. In a nonsimple action $A = (\{A_1, \ldots\}, R)$, a *chain rule* R is simply a sequential execution of the actions A_i in some order, but the order of the executions *does not matter*. That is, regardless of the order in which the subactions are executed, the result of A is achieved. A *coordination rule*, on the other hand, requires the execution of the actions A_i in some specific, non-arbitrary order, so that the order does matter.

We now formalize these notions. Let action A consist of a finite number of subactions, that is, $A = (\{A_1, A_2, \ldots, A_n\}, R)$. Given a permutation $\sigma = (i_1, i_2, \ldots, i_n)$ of the numbers $1, 2, \ldots, n$, the execution of the A_i according to σ is simply A_{i1} A_{i2} $\ldots A_{in.}$

In this notation, the rule R is a chain rule if the outcome of A is the same for all $n!$ permutations of the numbers $1, 2, \ldots, n$. That is, the outcome of the order of actions, $A_{i1}A_{i2} \ldots A_{in}$ is the same for all permutations $(i_1, i_2, \ldots, i_n)$ of $1,2,\ldots,n$. The rule R is a coordination rule if this is not the case; that is, if there exists at least one permutation $\tau = (j_1, j_2, \ldots, j_n)$ of the numbers $1, 2, \ldots, n$ so that the execution of the actions A_i according to τ, i.e., $A_{j1}A_{j2} \ldots A_{jn}$, is *not* the same as the outcome of the action A. Hence, the outcome of A_i is given by at least one, but *not all*, permutations of the A_i. This extends similarly to the cases where A consists of infinitely many actions.

These definitions are summarized as the first three action axioms, and refined in the following section.

(A1) There exists a simple action $\tilde{A}$.
(A2) Every action A is either simple (so $A = \tilde{A}$,) or composed of at least two previously defined actions $\{A_1, \ldots\}$ and a rule R for organizing those actions (so $A = (\{A_1, \ldots\}, R)$).
(A3) Each rule is either a chain or a coordination.

To motivate the definition of hierarchical complexity in the next section, we will rely on the following example.

Example 1. Let $+$ and $\times$ denote the traditional addition and multiplication on the real numbers, and let $\oplus$ and $\otimes$ denote the traditional addition and multiplication of variables (having values, say, in the real numbers). Then, consider the following four actions.

(a) $A = (\{+, \times\}, R_A)$ consisting of $1 + 2$ (i.e., adding the numbers 1 and 2) followed by 3×4 (i.e., multiplying the numbers 3 and 4). Clearly, the order in which the two subactions are executed does not matter: adding 1 and 2 and then multiplying 3 and 4 yields the same results, namely 3 and 12, as multiplying 3 and 4 and then adding 1 and 2. Thus, A is a chain action.

(b) $B = (\{+,\otimes\}, R_B)$ consisting of $1 + 2$ followed by $x \otimes y$. Again, the order in which the two subactions are executed does not matter: adding 1 and 2 and then multiplying x and y yields the same results, namely 3 and xy, as multiplying x and y and then adding 1 and 2. Thus, B is also a chain action.

(c) $C = (\{+, \times\}, R_C)$ consisting of the expression $2 \times (3 + 4)$. This is not a chain, for the order of the subactions matters: if we multiply 2 and 3 first and then add 4, we get 10, not 14, which is the answer dictated by rule R_C (i.e., adding 3 and 4 first and multiplying the result by 2). Thus C is a coordination, not a chain.

(d) $D = (\{\oplus, \otimes\}, R_D)$ consisting of the expression x $\otimes$ (1 $\oplus$ 2). Notice that because the expression involves real numbers *and* variables, we must necessarily use $\oplus$ and $\otimes$ and not simply $+$ and $\times$. In particular, because the distributive law dictates that x $\otimes$ (1 $\oplus$ 2) = (x $\otimes$ 1) $\oplus$ (x $\otimes$ 2), we cannot replace $\oplus$ by $+$. This observation will be important in the next section. As in the previous case, it is clear that D is a coordination action.

(e) $E = (\{\oplus, \otimes\}, R_E)$ consisting of the expression x $\otimes$ (y $\oplus$ z). This is exactly the same as (c) but at a more abstract level, and is, therefore, a coordination rule.

HIERARCHICAL COMPLEXITY

To each action A we wish to associate a notion of that action's hierarchical complexity, $h\ (A)$. Because actions are defined inductively, so is the function h, known as the order of the hierarchical complexity. For a simple action A, $h\ (A) = 0$. For a non-simple action, $A = (\{A_1, \ldots\}, R)$, we have to consider several cases. To get an intuitive idea, we analyze the complexity of the actions in Example 1.

Example 1 (Continued). Let m be the hierarchical complexity of $+$ and $\times$, the traditional addition and multiplication on the real numbers, and let n be the hierarchical complexity of the operations $\oplus$ and $\otimes$, the traditional addition and multiplication of variables. Intuitively we understand that $m < n$.

(a) Action A is a chain. The order in which the subactions forming the chain are executed can be changed without impacting the product of the actions. Therefore, executing A does not require any skill beyond the execution of each of the subactions individually.

(b) Similarly, B is a chain rule, but executing B requires being able to multiply at the abstract level (which is more complex than adding at the primary level), and so $h(B) = \max\ ((h(+), h(\otimes)) = h(\otimes) = n$. Notice that unlike action A, action B consists of subactions of different complexities.

(c) Observe now that action Ccoordinates two subactions of the same order, namely m. Because the order in which the two subactions are executed is nonarbitrary, the hierarchical complexity of this action is higher than the complexity of its subactions: $h(\mathrm{c}) > \max(h(+), h(\times)) = m$.

(d) As we remarked in Example 1, it may seem at first that action D coordinates two actions of different orders, $+$ of lower order and $\otimes$ of higher order. However, due to the distributive law, it actually coordinates two actions of the same order, that is, n. In particular, we observe that a coordinating action, at least in arithmetic, necessarily coordinates subactions of equal order. As in the previous case, we see that $h(D) > \max(h(\oplus), h(\otimes)) = n$.

(e) Lastly, as in (c), it is clear that $h(E) > \max\ (h\ (\oplus), h\ (\otimes)) = n$.

This analysis illustrates that the only way to raise hierarchical complexity is by coordinating actions of lower complexity. Moreover, coordination requires the subactions to be of equal orders. In light of Example 1, we now state the hierarchical complexity axioms that incorporate the action axioms (A1)–(A3).

Hierarchical Complexity Axioms

(HC1) There exists a simple action $\tilde{A}$, and $h(\tilde{A}) = 0$.

(HC2) Every nonsimple action $A = (\{A_1, \ldots\}, R)$ is either a chain of at least two previously defined actions of arbitrary orders of hierarchical complexity or a coordination of at least two previously defined actions *all of* which have the same order of hierarchical complexity.

(HC3) For a nonsimple action $A = (\{A_1, \ldots\}, R)$, $h(A) = \max_i h(\mathrm{A}_i)$ if A is a chain, and $h(A) = \max_i \mathrm{h}(A_1) + 1$ if A is a coordination.

Notice that by Axiom (HC2), a coordination action $A = (\{A_1, \ldots\}, R)$ necessarily coordinates subactions of equal orders of hierarchical complexity (i.e., $h(A1) = h(A2) = \ldots$), and so the order of hierarchical complexity of A is one higher than the order of hierarchical complexity of all its subactions. In particular, in the last equation in Axiom (HC3) we may replace A_1 by any subaction of A and still obtain the same result.

As a consequence of these axioms, we see that if we let $\boldsymbol{A}$ denote the collection of all actions in a given system, then the hierarchical complexity is a function $h\colon \boldsymbol{A} \to \boldsymbol{N}$, where $\boldsymbol{N} = \{0, 1, \ldots\}$ is the set of natural numbers (and zero) under the usual ordering.

Consequences of Hierarchical Complexity Axioms

1. (C1) (Discreteness) The order of hierarchical complexity of any action is a nonnegative integer. In particular, there are gaps between orders.
2. (C2) (Existence) If there exists an action of order n and an action of order $n+2$, then there necessarily exists an action of order $n + 1$.
3. (C3) (Comparison) For any two actions A and B, exactly one of the following holds: $h(A) > h(B)$, $h(A) = h(B)$, $h(A) < h(B)$. That is, the orders of hierarchical complexity of any two actions can be compared.

4. (C4) (Transitivity) For any three actions A, B, and C, if $h(A) > h(B)$ and $h(B) > h(C)$, then $h(A) > h(C)$.

In light of the descriptions of the orders of hierarchical complexity (see "Introduction to the Model of Hierarchical Complexity," this issue) for, among others, arithmetic tasks, we can assign the exact natural numbers corresponding to the orders of tasks in Example 1.

Example 1 (Continued). Both $+$ and $\times$ have order 7, that is, primary, whereas $\oplus$ and $\otimes$ have order 9, that is, abstract.

(a) Because A is a chain, $h(A) = \max\ (h(+), h(\times)) = 7$, that is, also primary.
(b) Because B is a chain, $h(B) = \max\ ((h(+), h(\otimes)) = 9$, that is, also abstract.
(c) Because C is a coordination, $h(C) = \max\ (h(+), h(\times)) + 1 = 8$, that is, concrete.
(d) Because D is a coordination, $h(D) = \max\ (h(\oplus), h(\otimes)) = 10$, that is, formal.
(e) Again, because E is a coordination, $h(E) = \max\ (h(\oplus), h(\otimes)) +1 = 10$, that is, formal.

Stages

The notion of stages is fundamental in the description of human, organismic, and machine evolution. Previously it has been defined in some ad hoc ways; here we describe it formally in terms of the Model of Hierarchical Complexity. Given a collection of actions $\boldsymbol{A}$ and a participant S performing $\boldsymbol{A}$, the *stage of performance* of S on $\boldsymbol{A}$ is the highest order of the actions in $\boldsymbol{A}$ completed successfully, that is, it is

$$\text{stage}(S, \boldsymbol{A}) = max\{h(A) | A \in \boldsymbol{A}\, and\ A \text{ completed successfully by } S\}. \qquad (1)$$

Thus, the notion of stage is discontinuous, having the same gaps as the orders of hierarchical complexity. This is in agreement with previous definitions (Commons, Trudeau, Stein, Richards, and Krause, 1998; Commons and Miller, 2001).

Measure of Hierarchical Complexity

We define the *measure of complexity at order n*, denoted by φ_n, as the minimum number of simple actions required to complete an action of order n. By axioms (HC2) and (HC3), an action of order n organizes at least two actions of order $n - 1$, each of which in turn organizes at least two actions of order $n - 2$, and so forth, until we reach the lowest-order, simple actions. Consequently, given the inductive definition of the hierarchical complexity orders, it is not surprising that $\varphi_n = 2^n$. Formally, a zero-order action consists of at least one simple action, so $\varphi_0 = 1 = 2^0$. For the inductive case, suppose $\varphi_{n-1} = 2^{n-1}$. Because by axioms (HC2) and (HC3), an action of order n is either a coordination of at least two actions of order $n - 1$ or a chain that includes an action of order n (and hence eventually is composed of at least two actions of order $n - 1$), we have $\varphi_n = 2\varphi_{n-1} = 2^n$, by induction.

CONCLUSION

This formal presentation of the theory, the Model of Hierarchical Complexity, supplemented with examples, establishes its purely mathematical basis. This underlies its universal applicability to any event that involves task-actions of any kind, that is, where there is activity at any scale, regardless of content. It makes a measurable distinction between traditional complexity and hierarchical complexity. It explains and measures the increases in hierarchical complexity identifiable in evolution from simple to complex organisms, human cognition, and human societies and their systems.

NOTE

1. Portions reprinted with permission. M. L. Commons, E. A. Goodheart, A. Pekker, T. L. Dawson, K. Draney, and K. M. Adams. 2007. Using Rasch scaled stage scores to validate orders of hierarchical complexity of balance beam task sequences, 124–130, and in *Rasch measurement: Advanced and specialized applications*, Eds. E. V. Smith, Jr. and R. M. Smith, 121–147. Maple Grove, MN: JAM Press. © 2007 by JAM Press.

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